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Generalized Gronwall-Bellman-type discrete inequalities and their applications

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Abstract

In this paper, some new nonlinear Gronwall-Bellman-type discrete inequalities are established, which can be used as a handy tool in the research of qualitative and quantitative properties of solutions of certain difference equations. The established results generalize some of the recent results obtained by Cheung and Ma, respectively.

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1 Introduction

In the past few years, the research of Gronwall-Bellman-type finite difference inequalities has been payed much attention by many authors, which play an important role in the study of qualitative as well as quantitative properties of solutions of difference equations, such as boundedness, stability, existence, uniqueness, continuous dependence and so on. Many difference inequalities have been established (see [1]-[11] and the references therein). But in the analysis of some certain difference equations, the bounds provided by the earlier inequalities are inadequate and it is necessary to seek some new discrete inequalities in order to obtain a diversity of desired results. Our aim in this paper is to establish some new nonlinear Gronwall-Bellman-type discrete inequalities, which provide new bounds for unknown functions lying in these inequalities. We will illustrate the usefulness of the established results by applying them to study the boundedness, uniqueness, and continuous dependence on initial data of solutions of certain difference equations. Our results generalize some of the results in [1,2].

Throughout this paper, $\mathbf{R}$ denotes the set of real numbers and $\mathbf{R}_+ = [0, \infty)$, while $\mathbf{Z}$ denotes the set of integers. The definition domain and the image of a function f are denoted by $Dom(f)$ and $Im(f)$, respectively. $I := [m_0, \infty) \cap \mathbf{Z}$ and $J := [n_0, \infty) \cap \mathbf{Z}$ are two fixed lattices of integral points in $\mathbf{R}$. Let $\Omega := I \times J \subset \mathbf{Z}^2$, and $\wp(\Omega)$ denotes the set of all $\mathbf{R}$-valued functions on Ω, while $\wp_+(\Omega)$ denotes the set of all $\mathbf{R}_+$-valued functions on Ω. For the sake of convenience, we extend the domain of definition of each function in $\wp(\Omega)$ and $\wp_+(\Omega)$ trivially to the ambient space $\mathbf{Z}^2$. So, for example, a function in $\wp(\Omega)$ is regarded as a function defined on $\mathbf{Z}^2$ with support in $\wp(\Omega)$. As usual, the collection of all continuous functions of a topological space X into a topological space Y will be denoted by $C(X, Y)$. Finally, the partial difference operators Δ_1 and Δ_2 on u

$\in \wp(\mathbf{Z}^2)$ are defined as $\Delta_1 u(m, n) = u(m + 1, n) - u(m, n)$, $\Delta_2 u(m, n) = u(m, n + 1) - u(m, n)$.

2 Main results

Theorem 1 *Let $\Omega_{(s,t)} = ([m_0, s] \times [n_0, t]) \cap \Omega$, $(s, t) \in \Omega$. Suppose $u(m, n)$, $a(m, n)$, $k(m, n)$, $b(m, n) \in \wp_+(\Omega)$, and a, k are nondecreasing in every variable. η, $\phi \in C(\mathbf{R}_+, \mathbf{R}_+)$, and η are strictly increasing, while ϕ is nondecreasing with $\phi(r) > 0$ for $r > 0$. If for $(m, n) \in \Omega$, $u(m, n)$, satisfies the following inequality*

$$\eta(u(m,n)) \le a(m,n) + k(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\varphi(u(s,t)), \tag{1}$$

then for $(m,n) \in \Omega_{(m_1,n_1)}$, we have

$$u(m,n) \le \eta^{-1}\left\{G^{-1}\left[G(a(m,n)) + k(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\right]\right\}, \tag{2}$$

where

$$G(z) = \int_{z_0}^{z} \frac{1}{\varphi(\eta^{-1}(z))}dz, z \ge z_0 > 0, \tag{3}$$

and m_1, n_1 are chosen so that for $(m,n) \in \Omega_{(m_1,n_1)}$,

$$G(a(m,n)) + k(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t) \in Dom(G^{-1}). \tag{4}$$

Proof Given $(X,Y) \in \Omega_{(m_1,n_1)}$, and let $(m, n) \in \Omega_{(X,Y)}$. Then considering a, k are both nondecreasing, from (1), we have

$$\eta(u(m,n)) \le a(X,Y) + k(X,Y)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\varphi(u(s,t)),\ (m,n) \in \Omega_{(X,Y)} \tag{5}$$

Let the right side of (5) be $v(m, n)$. Then

$$u(m,n) \le \eta^{-1}(v(m,n)),\ (m,n) \in \Omega_{(X,Y)}, \tag{6}$$

and

$$\begin{aligned}\Delta_1 v(m,n) &= v(m+1,n) - v(m,n) = k(X,Y)\sum_{t=n_0}^{n-1} b(m,t)\varphi(u(m,t))\\ &\le k(X,Y)\sum_{t=n_0}^{n-1} b(m,t)\varphi\left(\eta^{-1}(v(m,t))\right) \le k(X,Y)\varphi\left(\eta^{-1}(v(m,n-1))\right)\sum_{t=n_0}^{n-1} b(m,t).\end{aligned} \tag{7}$$

On the other hand, according to the Mean-Value Theorem for integrals, there exists ξ such that $v(m, n) \le \xi \le v(m + 1, n)$, and

$$\begin{aligned}\Delta_1 G(v(m,n)) &= G(v(m+1,n)) - G(v(m,n))\\ &= \int_{v(m,n)}^{v(m+1,n)} \frac{1}{\varphi(\eta^{-1}(z))}dz = \frac{\Delta_1 v(m,n)}{\varphi(\eta^{-1}(\xi))} \le \frac{\Delta_1 v(m,n)}{\varphi(\eta^{-1}(v(m,n)))}.\end{aligned} \tag{8}$$

Combining (7) and (8), we obtain

$$\Delta_1 G(v(m,n)) \le k(X,Y)\frac{\varphi\left(\eta^{-1}(v(m,n-1))\right)}{\varphi\left(\eta^{-1}(v(m,n))\right)}\sum_{t=n_0}^{n-1} b(m,t) \le k(X,Y)\sum_{t=n_0}^{n-1} b(m,t). \quad (9)$$

Setting $m = s$, and a summary on both sides of (9) with respect to s from m_0 to m - 1 yields

$$\sum_{s=m_0}^{m-1} \Delta_1 G(v(s,n)) \le k(X,Y)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t), \quad (10)$$

that is,

$$G(v(m,n)) - G(v(m_0,n)) \le k(X,Y)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t). \quad (11)$$

Considering G is strictly increasing, and $v(m_0, n) = a(X, Y)$, it follows

$$v(m,n) \le G^{-1}\left[G(a(X,Y)) + k(X,Y)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\right], \quad (m,n) \in \Omega_{(X,Y)}. \quad (12)$$

Combining (6) and (12), we have

$$u(m,n) \le \eta^{-1}\left\{G^{-1}\left[G(a(X,Y)) + k(X,Y)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\right]\right\}, \quad (m,n) \in \Omega_{(X,Y)}. \quad (13)$$

Take $m = X$, $n = Y$ in (13), yields

$$u(X,Y) \le \eta^{-1}\left\{G^{-1}\left[G(a(X,Y)) + k(X,Y)\sum_{s=m_0}^{X-1}\sum_{t=n_0}^{Y-1} b(s,t)\right]\right\}. \quad (14)$$

Since $(X,Y) \in \Omega_{(m_1,n_1)}$ are selected arbitrarily, then in fact (14) holds for $\forall (m,n) \in \Omega_{(m_1,n_1)}$, that is

$$u(m,n) \le \eta^{-1}\left\{G^{-1}\left[G(a(m,n)) + k(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\right]\right\}, \quad (m,n) \in \Omega_{(m_1,n_1)},$$

which is the desired result. □

Remark 1 If we take $a(m, n) \equiv c$, $k(m, n) \equiv 1$, $\eta(u) = u^\alpha$ in Theorem 2.1, where $c \ge 0$, $\alpha > 0$ are constants, then Theorem 2.1 reduces to [1, Theorem 2.1].

Corollary 1 *Under the conditions of Theorem 2.1, if for* $(m, n) \in \Omega$, $u(m, n)$ *satisfies the following inequality*

$$\eta(u(m,n)) \le a(m,n) + k(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\eta(u(s,t)), \quad (15)$$

then for $(m,n) \in \Omega_{(m_1,n_1)}$, *we have*

$$u(m,n) \le \eta^{-1}\left(a(m,n)e^{J(m,n)}\right), \quad (16)$$

where

$$J(m,n) = k(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t). \tag{17}$$

Remark 2 In Corollary 2.2, if we take η for different functions, then we have various bounds for $u(m, n)$. For example, if we take $\eta(u) = u$, then we obtain $u(m, n) \leq a(m, n)e^{J(m, n)}$, while $u(m,n) \leq C^{\frac{1}{p}} e^{\frac{J(m,n)}{p}}$, if we take $\eta(u) = u^p$, $a(m, n) \equiv C$, $k(m, n) \equiv 1$.

Following in a same manner as the proof of Theorem 2.1, we can obtain the following three theorems.

Theorem 2 *Let* $\Omega_{(s,\ t)} = ([m_0, s] \times [t, \infty)) \cap \Omega$, $(s, t) \in \Omega$. *Suppose* u, a, k, b, η, ϕ *are defined the same as in Theorem 2.1. If for* $(m, n) \in \Omega$, $u(m, n)$ *satisfies the following inequality*

$$\eta(u(m,n)) \leq a(m,n) + k(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n+1}^{\infty} b(s,t)\varphi(u(s,t)), \tag{18}$$

then for $(m,n) \in \Omega_{(m_1,n_1)}$, *we have*

$$u(m,n) \leq \eta^{-1}\left\{G^{-1}\left[G(a(m,n)) + k(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n+1}^{\infty} b(s,t)\right]\right\}, \tag{19}$$

where G is defined as in Theorem 2.1, and m_1, n_1 *are chosen so that for* $G(a(m,n)) + k(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n+1}^{\infty} b(s,t) \in Dom(G^{-1})$,
$G(a(m,n)) + k(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n+1}^{\infty} b(s,t) \in Dom(G^{-1})$.

Theorem 3 *Let* $\Omega_{(s,\ t)} = ([s, \infty) \times [n_0, t]) \cap \Omega$, $(s, t) \in \Omega$. *Suppose* u, a, k, b, η, ϕ *are defined the same as in Theorem 2.1. If for* $(m, n) \in \Omega$, $u(m, n)$ *satisfies the following inequality*

$$\eta(u(m,n)) \leq a(m,n) + k(m,n)\sum_{s=m+1}^{\infty}\sum_{t=n_0}^{n-1} b(s,t)\varphi(u(s,t)), \tag{20}$$

then for $(m,n) \in \Omega_{(m_1,n_1)}$, *we have*

$$u(m,n) \leq \eta^{-1}\left\{G^{-1}\left[G(a(m,n)) + k(m,n)\sum_{s=m+1}^{\infty}\sum_{t=n_0}^{n-1} b(s,t)\right]\right\}, \tag{21}$$

where G is defined as in Theorem 2.1, and m_1, n_1 *are chosen so that for* $G(a(m,n)) + k(m,n)\sum_{s=m+1}^{\infty}\sum_{t=n_0}^{n-1} b(s,t) \in Dom(G^{-1})$,
$G(a(m,n)) + k(m,n)\sum_{s=m+1}^{\infty}\sum_{t=n_0}^{n-1} b(s,t) \in Dom(G^{-1})$.

Theorem 4 *Let* $\Omega_{(s,\ t)} = ([s, \infty) \times [t, \infty)) \cap \Omega$, $(s, t) \in \Omega$. *Suppose* u, a, k, b, η, ϕ *are defined the same as in Theorem 2.1. If for* $(m, n) \in \Omega$, $u(m, n)$ *satisfies the following inequality*

$$\eta(u(m,n)) \leq a(m,n) + k(m,n)\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} b(s,t)\varphi(u(s,t)), \tag{22}$$

then for $(m,n) \in \Omega_{(m_1,n_1)}$*, we have*

$$u(m,n) \le \eta^{-1}\left\{G^{-1}\left[G(a(m,n)) + k(m,n)\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} b(s,t)\right]\right\}, \tag{23}$$

where G is defined as in Theorem 2.1, and m_1*,* n_1 *are chosen so that for* $G(a(m,n)) + k(m,n)\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} b(s,t) \in Dom(G^{-1})$*,*
$G(a(m,n)) + k(m,n)\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} b(s,t) \in Dom(G^{-1})$*.*

In the following theorem, we will study a class of Volterra-Fredholm type difference inequality.

Theorem 5 *Suppose* u*,* η*,* ϕ *are defined as in Theorem 2.1,* $a \in \wp_+(\Omega)$*,* M*,* N*,* C *are constants, and* $M \in [m_0, \infty) \cap \mathbf{Z}$*,* $N \in [n_0, \infty) \cap \mathbf{Z}$*,* $C \ge 0$*. If for* $(m, n) \in \Omega$*,* $u(m, n)$ *satisfies the following inequality*

$$\eta(u(m,n)) \le C + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\varphi(u(s,t)) + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} a(s,t)\varphi(u(s,t)), \tag{24}$$

then we have

$$u(m,n) \le \eta^{-1}\left\{G^{-1}\left\{G\left(\widetilde{G}^{-1}\left[\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} a(s,t)\right]\right) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\right\}\right\}, \quad (m,n) \in \Omega, \tag{25}$$

provided that $\widetilde{G}(z) = G(2z - C) - G(z)$ *is strictly increasing for* $z \ge C$*, where G is defined as in (3).*

Proof Suppose $C > 0$, and let the right side of (24) be $v(m, n)$. Then

$$u(m,n) \le \eta^{-1}(v(m,n)), \; (m,n) \in \Omega, \tag{26}$$

and

$$v(m_0,n) = C + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} a(s,t)\varphi(u(s,t)). \tag{27}$$

Furthermore,

$$\begin{aligned} \Delta_1 v(m,n) &= \sum_{t=n_0}^{n-1} a(m,t)\varphi(u(m,t)) \\ &\le \sum_{t=n_0}^{n-1} a(m,t)\varphi(\eta^{-1}(v(m,t))) \le \varphi(\eta^{-1}(v(m,n-1)))\sum_{t=n_0}^{n-1} a(m,t). \end{aligned} \tag{28}$$

On the other hand, according to the Mean-Value Theorem for integrals, there exists ξ such that $v(m, n) \le \xi \le v(m + 1, n)$, and

$$\begin{aligned} \Delta_1 G(v(m,n)) &= G(v(m+1,n)) - G(v(m,n)) \\ &= \int_{v(m,n)}^{v(m+1,n)} \frac{1}{\varphi(\eta^{-1}(z))}dz = \frac{\Delta_1 v(m,n)}{\varphi(\eta^{-1}(\xi))} \le \frac{\Delta_1 v(m,n)}{\varphi(\eta^{-1}(v(m,n)))}. \end{aligned} \tag{29}$$

Combining (28) and (29), we obtain

$$\Delta_1 G(v(m,n)) \le \frac{\varphi(\eta^{-1}(v(m,n-1)))}{\varphi(\eta^{-1}(v(m,n)))} \sum_{t=n_0}^{n-1} a(m,t) \le \sum_{t=n_0}^{n-1} a(m,t). \tag{30}$$

Setting $m = s$, and a summary on both sides of (30) with respect to s from m_0 to m - 1 yields

$$\sum_{s=m_0}^{m-1} \Delta_1 G(v(s,n)) \le \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t), \tag{31}$$

that is,

$$G(v(m,n)) - G(v(m_0,n)) \le \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t). \tag{32}$$

Considering G is strictly increasing, then it follows

$$v(m,n) \le G^{-1}\left[G(v(m_0,n)) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)\right], \quad (m,n) \in \Omega. \tag{33}$$

Take $m = M$, $n = N$ in (33), we obtain

$$v(M,N) \le G^{-1}\left[G(v(m_0,N)) + \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} a(s,t)\right]. \tag{34}$$

So,

$$\begin{aligned} 2v(m_0,n) - C &= C + 2\sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} a(s,t)\varphi(u(s,t)) = v(M,N) \\ &\le G^{-1}\left[G(v(m_0,N)) + \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} a(s,t)\right] = G^{-1}\left[G(v(m_0,n)) + \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} a(s,t)\right], \end{aligned} \tag{35}$$

that is,

$$G(2v(m_0,n) - C) - G(v(m_0,n)) \le \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} a(s,t), \tag{36}$$

which is rewritten as

$$\widetilde{G}(v(m_0,n)) \le \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} a(s,t). \tag{37}$$

Since $\widetilde{G}$ is strictly increasing, then furthermore we have

$$v(m_0,n) \le \widetilde{G}^{-1}\left[\sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} a(s,t)\right]. \tag{38}$$

Combining (26), (33), and (38), we obtain the desired inequality.

If $C = 0$, we can substitute C with $\varepsilon > 0$ in the proof above and then after letting $\varepsilon \to 0$, we obtain the desired result. □

Remark 3 If we take $\eta(u) = u$ in Theorem 2.6, then Theorem 2.6 reduces to [2, Theorem 2.1].

Corollary 2 *Under the conditions of Theorem 2.6, if* $C > 0$, *and for* $(m, n) \in \Omega$, $u(m, n)$ *satisfies the following inequality*

$$\eta(u(m,n)) \leq C + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\eta(u(s,t)) + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} a(s,t)\eta(u(s,t)), \tag{39}$$

then we have

$$u(m,n) \leq \eta^{-1}\left\{\frac{C}{2-e^{J(M,N)}}e^{J(m,n)}\right\}, \quad (m,n) \in \Omega, \tag{40}$$

provided that $e^{J(M,\ N)} < 2$, *where* $J(m,n) = \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)$.

Proof From Theorem 2.6, considering $\eta = \phi$, we have

$$G(z) = \int_{z_0}^{z}\frac{1}{z}dz = lnz - lnz_0 \tag{41}$$

and

$$\widetilde{G}(z) = G(2z - C) - G(z) = ln(2 - \frac{C}{z}). \tag{42}$$

Combining (25), (41) and (42), we can deduce the desired result. □

Remark 4 In Corollary 2.7, if we take η for different functions, then we have various bounds foe $u(m, n)$. For example, if we take $\eta(u) = u$, then we obtain $u(m,n) \leq \frac{C}{2-e^{J(M,N)}}e^{J(m,n)}$. If we take $\eta(u) = u^p$, then we obtain $u(m,n) \leq \{\frac{C}{2-e^{J(M,N)}}e^{J(m,n)}\}^{\frac{1}{p}}$.

Theorem 6 *Suppose* u, η, ϕ, a, C, G *are defined as in Theorem 2.6. Define* $\widetilde{J}(z) = G(2z - C) - G(z) - \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} a(s,t)$. *Furthermore, assume* $\widetilde{J}$ *is increasing, and* $\exists\xi \geq C$ *such that* $\widetilde{J}(\xi) = 0$. *If for* $(m, n) \in \Omega$, $u(m, n)$ *satisfies (24), then*

$$u(m,n) \leq \eta^{-1}\left\{G^{-1}\left[G(\xi) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\right]\right\}, \quad (m,n) \in \Omega. \tag{43}$$

Proof Let the right side of (24) be $v(m, n)$. Then

$$u(m,n) \leq \eta^{-1}(v(m,n)), \ (m,n) \in \Omega. \tag{44}$$

Similar to the process of (26)-(32), we deduce

$$v(m,n) \leq G^{-1}\left[G(v(m_0,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\right], \quad (m,n) \in \Omega, \tag{45}$$

and

$$G(2v(m_0,n) - C) - G(v(m_0,n)) \leq \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} a(s,t), \tag{46}$$

that is, $\widetilde{J}(v(m_0,n)) \leq 0$. Considering $\widetilde{J}(\xi) = 0$, then $\widetilde{J}(v(m_0,n)) \leq \widetilde{J}(\xi)$. Since $\widetilde{J}$ is increasing, then

$$v(m_0, n) \leq \xi. \tag{47}$$

Combining (45) and (47), we obtain

$$v(m,n) \leq G^{-1}\left[G(\xi) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\right], \quad (m,n) \in \Omega. \tag{48}$$

Then by (44) and (48), we obtain the desired inequality. □

Remark 5 If we take $\eta(u) = u$ in Theorem 2.8, then Theorem 2.8 reduces to [2, Theorem 2.1'].

3 Applications

In this section, we will present some applications for the established results above, and show that they are useful in the study of boundedness, uniqueness, and continuous dependence of solutions of certain difference equations.

Example 1 Consider the following difference equation

$$\Delta_{12} e^{u(m,n)} = F(m, n, u(m, n)) \tag{49}$$

with the initial condition

$$u(m, n_0) = ln\left[e^{f(m)} + 1\right], u(m_0, n) = ln\left[e^{g(n)} + 1\right], f(m_0) = g(n_0) = 0, \tag{50}$$

where $u \in \wp_+(\Omega)$, $F : \Omega \times \mathbf{R}_+ \to \mathbf{R}$, $f : I \to \mathbf{R}$, $g : J \to \mathbf{R}$.

Theorem 7 *Suppose $u(m, n)$ is a solution of (49) and (50), and $|F(m, n, u)| \leq b(m, n) u^p$, $e^{f(m)} + e^{g(n)} \leq C$, where p, C are constants, and $p > 1$, $C > 0$, $b \in \wp_+(\Omega)$, then for $(m, n) \in \Omega_{(m_1,n_1)}$, we have*

$$u(m,n) \leq ln\left\{G^{-1}\left[G(C) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\right]\right\}, \tag{51}$$

where

$$G(z) = \int_{z_0}^{z} \frac{1}{(lnz)^p} dz, z \geq z_0 > 0, \tag{52}$$

$\Omega_{(m_1,n_1)} = ([m_0, m_1] \times [n_0, n_1]) \bigcap \Omega$, *and* m_1, n_1 *are chosen so that for* $G(C) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t) \in Dom(G^{-1})$, $G(C) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t) \in Dom(G^{-1})$.

Proof The equivalent form of (49) and (50) can be denoted by

$$e^{u(m,n)} = e^{f(m)} + e^{g(n)} + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} F(s, t, u(s,t)). \tag{53}$$

So,

$$e^{u(m,n)} \leq e^{f(m)} + e^{g(n)} + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} |F(s,t,u(s,t))| \leq C + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)u^p(s,t). \tag{54}$$

Then, a suitable application of Theorem 2.1 (with $\eta(u) = e^{\alpha}$, $\phi(u) = u^p$) to (54) yields the desired result. □

Lemma 1 *For any* u_1, $u_2 \in \mathbf{R}^+$, *we have*

$$|u_1 - u_2| \leq |e^{u_1} - e^{u_2}|. \tag{55}$$

Proof Without loss of generality, we assume $u_1 \geq u_2$, and let u_2 be fixed. Define $f(z) = (e^z - e^{u_2}) - (z - u_2)$. Then we have $f'(z) = e_z - 1 \geq o$ for $z \geq u_2 \geq 0$, which shows $f(z)$ is nondecreasing on $[u_2, \infty)$. So $f(u_1) \geq f(u_2) = 0$, that is, $e^{u_1} - e^{u_2} \geq u_1 - u_2$, which shows $|u_1 - u_2| \leq |e^{u_1} - e^{u_2}|$. □

Theorem 8 *Assume* u_1, u_2 *are two solutions of (49) and (50), and* $|F(m, n, u_1) - F(m, n, u_2)| \leq b(m, n)|u_1 - u_2|$, $\forall (m, n) \in \Omega$, *where* $b \in \wp_+(\Omega)$, *then* $u_1 \equiv u_2$, *that is, (49) and (50) has at most one solution.*

Proof Since u_1, u_2 are two solutions of (49) and (50), then from (49), (50), and (53), we have

$$e^{u_1(m,n)} = e^{f(m)} + e^{g(n)} + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} F(s, t, u_1(s, t)) \tag{56}$$

and

$$e^{u_2(m,n)} = e^{f(m)} + e^{g(n)} + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} F(s, t, u_2(s, t)). \tag{57}$$

By (56) and (57) and Lemma 3.2, we deduce

$$\begin{aligned}
&\left|e^{u_1(m,n)} - e^{u_2(m,n)}\right| \\
&= \left|\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} [F(s, t, u_1(s, t)) - F(s, t, u_2(s, t))]\right| \\
&\leq \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} |F(s, t, u_1(s, t)) - F(s, t, u_2(s, t))| \\
&\leq \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s, t)|u_1(s, t) - u_2(s, t)| \\
&\leq \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s, t)|e^{u_1(s,t)} - e^{u_2(s,t)}|.
\end{aligned} \tag{58}$$

A suitable application of Corollary 2.2 to (58) yields $|e^{u_1(m,n)} - e^{u_2(m,n)}| \leq 0$, which implies $u_1 \equiv u_2$, and the proof is complete.

The following theorem deals with the continuous dependence of the solution of (49) and (50) on the function F and the initial value $f(m)$, $g(n)$. □

Theorem 9 *Assume* $u(m, n)$ *is the solution of (49) and (50),* $|F(m, n, u_1) - F(m, n, u_2)| \leq b(m, n)|u_1 - u_2|$, $\forall (m, n) \in \Omega$, *where* $b \in \wp_+(\Omega)$, $|e^{f(m)} - e^{\bar{f}(m)} + e^{g(n)} - e^{\bar{g}(n)}| \leq \varepsilon$, *where* $\varepsilon > 0$ *is a constant. Furthermore, assume* $\bar{u} \in \wp_+(\Omega)$, $\bar{u} \in \wp_+(\Omega)$*is the solution of the following difference equation*

$$\Delta_{12} e^{\bar{u}(m,n)} = \bar{F}(m, n, \bar{u}(m, n)) \tag{59}$$

with the initial condition

$$\bar{u}(m,n_0)=ln[e^{\bar{f}(m)}+1],\ \bar{u}(m_0,n)=ln[e^{\bar{g}(n)}+1],\ \bar{f}(m_0)=\bar{g}(n_0)=0, \tag{60}$$

where $\bar{F}:\Omega\times\mathbf{R}_+\to\mathbf{R}$, $\bar{f}:I\to\mathbf{R}$, $\bar{g}:J\to\mathbf{R}$, *then*

$$|u(m,n)-\bar{u}(m,n)|\ \le 2\varepsilon e^{J(m,n)}, \tag{61}$$

where

$$J(m,n)=\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}b(s,t). \tag{62}$$

Proof From (59) and (60), we deduce

$$e^{\bar{u}(m,n)}=e^{\bar{f}(m)}+e^{\bar{g}(n)}+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\bar{F}(s,t,\bar{u}(s,t)). \tag{63}$$

Then by a combination of (53) and (63), we deduce

$$\begin{aligned}
&\left|e^{u(m,n)}-e^{\bar{u}(m,n)}\right|\\
&\le\left|e^{f(m)}-e^{\bar{f}(m)}+e^{g(n)}-e^{\bar{g}(n)}\right|+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}|F(s,t,u(s,t))-\bar{F}(s,t,\bar{u}(s,t))|\\
&\le\varepsilon+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}|F(s,t,u(s,t))-F(s,t,\bar{u}(s,t))+F(s,t,\bar{u}(s,t))-\bar{F}(s,t,\bar{u}(s,t))|\\
&\le\varepsilon+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[|F(s,t,u(s,t))-F(s,t,\bar{u}(s,t))|+|F(s,t,\bar{u}(s,t))-\bar{F}(s,t,\bar{u}(s,t))|]\\
&\le 2\varepsilon+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}|F(s,t,u(s,t))-F(s,t,\bar{u}(s,t))|\\
&\le 2\varepsilon+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}b(s,t)|u(s,t)-\bar{u}(s,t)|\\
&\le 2\varepsilon+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}b(s,t)|e^{u(s,t)}-e^{\bar{u}(s,t)}|.
\end{aligned} \tag{64}$$

After a suitable application of Corollary 2.2 to (64), we obtain $|e^{u(m,n)}-e^{\bar{u}(m,n)}|\ \le 2\varepsilon e^{J(m,n)}$. So from Lemma 3.2, it follows $|u(m,n)-\bar{u}(m,n)|\ \le\ |e^{u(m,n)}-e^{\bar{u}(m,n)}|\ \le 2\varepsilon e^{J(m,n)}$, which is the desired result. □

Example 2 Consider the following Volterra-Fredholm sum-difference equation

$$u^p(m,n)=f^p(m)+g^p(n)+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}F(s,t,u(s,t))+\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}F(s,t,u(s,t)) \tag{65}$$

with the initial condition

$$u(m,n_0)=f(m),u(m_0,n)=g(n),f(m_0)=g(n_0)=0, \tag{66}$$

where $u\in\wp(\Omega)$, $F:\Omega\times\mathbf{R}\to\mathbf{R}$, $f:\mathrm{I}\to\mathbf{R}$, $g:J\to\mathbf{R}$, $p\ge 1$ is an odd number.

Theorem 10 *Suppose* $u(m,n)$ *is a solution of (65) and (66), and* $|F(m,n,u)|\le a(m,n)\,|u|^p$, $|f^p(m)+g^p(n)|\le C$, *where* $C>0$ *is a constant, and* $a\in\wp_+(\Omega)$, *then we have*

$$|u(m,n)| \leq \left\{ \frac{C}{2-e^{J(M,N)}} e^{J(m,n)} \right\}^{\frac{1}{p}} \tag{67}$$

provided $e^{J(M,\ N)} < 2$, *where*

$$J(m,n) = \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t). \tag{68}$$

Proof From (65) and (66), we have

$$\begin{aligned} |u(m,n)|^p &\leq |f^p(m)+g^p(n)| + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} |F(s,t,u(s,t))| + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} |F(s,t,u(s,t))| \\ &\leq C + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)|u(s,t)|^p + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} a(s,t)|u(s,t)|^p \end{aligned} \tag{69}$$

Then a suitable application of Corollary 2.7 to (69) yields the desired result. □

Theorem 11 *Assume* $|F(m,n,u_1) - F(m,n,u_2)| \leq a(m,n)|u_1^p - u_2^p|$, $\forall (m,\ n) \in \ \Omega$, *where* $a \in \wp_+(\Omega)$, *and* $e^{J(M,\ N)} < 2$, *where* $J(m,n) = \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)$, *then (65) and (66) has at most one solution.*

Proof Suppose u_1, u_2 are two solutions of (65) and (66), then from (65) and (66), we have

$$\begin{aligned} &\left| u_1^p(m,n) - u_2^p(m,n) \right| \\ &= \left| \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} [F_1(s,t,u(s,t)) - F_2(s,t,u(s,t))] + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} [F_1(s,t,u(s,t)) - F_2(s,t,u(s,t))] \right| \\ &\leq \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} |F_1(s,t,u(s,t)) - F_2(s,t,u(s,t))| + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} |F_1(s,t,u(s,t)) - F_2(s,t,u(s,t))| \\ &\leq \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)|u_1^p(s,t) - u_2^p(s,t)| + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} a(s,t)|u_1^p(s,t) - u_2^p(s,t)|. \end{aligned} \tag{70}$$

A suitable application of Corollary 2.7 to (70) yields $|u_1^p(m,n) - u_2^p(m,n)| \leq 0$, that is, $u_1^p(m,n) = u_2^p(m,n)$, $\forall (m,\ n) \in \ \Omega$. Since p is an odd number, then $u_1(m,\ n) = u_2(m,\ n)$, which implies (65) and (66) has at most one solution.

Similar to Theorem 3.4, we have the following result showing the continuous dependence of the solution of (65) and (66) on the function F and the initial data $f(m)$, $g(n)$. □

Theorem 12 *Assume* $u(m,\ n)$ *is the solution of (65) and (66),* $|F(m,n,u_1) - F(m,n,u_2)| \leq a(m,n)|u_1^p - u_2^p|$, $\forall (m,\ n) \in \ \Omega$, *where* $a \in \wp_+(\Omega)$, $|f^p(m) - \bar{f}^p(m) + g^p(n) - \bar{g}^p(n)| \leq \varepsilon$, *where* $\varepsilon > 0$ *is a constant. Furthermore, assume* $\bar{u} \in \wp_+(\Omega)$, $\bar{u} \in \wp_+(\Omega)$*is the solution of the following difference equation*

$$\bar{u}^p(m,n) = \bar{f}^p(m) + \bar{g}^p(n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \bar{F}(s,t,\bar{u}(s,t)) + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} \bar{F}(s,t,\bar{u}(s,t)) \tag{71}$$

with the initial condition

$$\bar{u}(m,n_0) = \bar{f}(m), \bar{u}(m_0,n) = \bar{g}(n), \bar{f}(m_0) = \bar{g}(n_0) = 0, \tag{72}$$

where $\bar{F}: \Omega \times \mathbf{R} \to \mathbf{R}$, $\bar{f}: I \to \mathbf{R}$, $\bar{g}: J \to \mathbf{R}$, *then*

$$|u^p(m,n) - \bar{u}^p(m,n)| \leq \frac{2\varepsilon}{2 - e^{J(M,N)}} e^{J(m,n)}, \quad (m,n) \in \Omega, \tag{73}$$

provided that $e^{J(M,\,N)} < 2$, *where* $J(m,n) = \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)$.

The proof for Theorem 3.7 is similar to Theorem 3.4, and we omit it here.

4 Competing interests

The authors declare that they have no competing interests.

5 Authors' contributions

QF carried out the main part of this article. All authors read and approved the final manuscript.

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